Model Reduction of Linear Conservative Mechanical Systems

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(3.8), and (3.9), we can see that

$$\mathcal{P}_{\text{closed loop with min. controller}} + \mathcal{P}_{\text{removal}} = \mathcal{P}_{\text{regulator}} + \mathcal{P}_{\text{observer}} + \mathcal{P}_{K(s)}. \tag{3.10}$$

To complete the proof, we need to show that $\mathcal{P}_{\text{removal}}$ is a subset of $\mathcal{P}_{\text{regulator}} + \mathcal{P}_{\text{observer}}$ when $K(s)$ is minimal. The dynamic equations of the observer-based controller in Fig. 4, i.e., the block diagram inside the dotted-line box, can be written as follows:

$$\dot{\hat{x}} = (A + B_2F + HC_2 + HD_{22}F)\hat{x} + [-H \quad -(B_2 + HD_{22})] \begin{bmatrix} y \\ u_2 \end{bmatrix} \tag{3.11a}$$

$$\begin{bmatrix} u \\ \hat{y} \end{bmatrix} = \begin{bmatrix} F \\ -(C_2 + D_{22}F) \end{bmatrix} \hat{x} + \begin{bmatrix} 0 & -I \\ I & D_{22} \end{bmatrix} \begin{bmatrix} y \\ u_2 \end{bmatrix}. \tag{3.11b}$$

Assume that the added dynamics $K(s)$ is described by the following *minimal* relation:

$$\dot{k} = \hat{A}k + \hat{B}\hat{y} \tag{3.12a}$$

$$u_2 = \hat{C}k + \hat{D}\hat{y}. \tag{3.12b}$$

The controller $Q(s)$ is just a combination of (3.11) and (3.12). From (3.11) and (3.12), we have the dynamic equations of the controller $Q(s)$ as follows:

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{k} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \hat{x} \\ k \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} y \tag{3.13a}$$

$$u = [\gamma_1 \quad \gamma_2] \begin{bmatrix} \hat{x} \\ k \end{bmatrix} + \delta y \tag{3.13b}$$

where

$$\beta_1 = -H - (B_2 + HD_{22})(I - \hat{D}D_{22})^{-1}\hat{D} \tag{3.14a}$$

$$\beta_2 = \hat{B} + \hat{B}D_{22}(I - \hat{D}D_{22})^{-1}\hat{D} \tag{3.14b}$$

$$\gamma_1 = F + (I - \hat{D}D_{22})^{-1}\hat{D}(C_2 + D_{22}F) \tag{3.14c}$$

$$\gamma_2 = -(I - \hat{D}D_{22})^{-1}\hat{C} \tag{3.14d}$$

$$\alpha_{11} = A + HC_2 + (B_2 + HD_{22})\gamma_1 \tag{3.14e}$$

$$= A + B_2F - \beta_1(C_2 + D_{22}F) \tag{3.14f}$$

$$\alpha_{12} = (B_2 + HD_{22})\gamma_2 \tag{3.14g}$$

$$\alpha_{21} = -\beta_2(C_2 + D_{22}F) \tag{3.14h}$$

$$\alpha_{22} = \hat{A} - \hat{B}D_{22}\gamma_2 \tag{3.14i}$$

$$\delta = -(I - \hat{D}D_{22})^{-1}\hat{D}. \tag{3.14j}$$

Now, assume that the state-space representation (3.13) of the controller $Q(s)$ is unobservable. Then by the PBH test [6], there exists a nonzero vector ξ such that

$$[\gamma_1 \quad \gamma_2] \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = 0; \quad \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \xi \tag{3.15a}$$

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \lambda \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \tag{3.15b}$$

for some eigenvalue λ of (3.13). Note that it is the eigenvalue λ that is unobservable. From (3.15b), we get

$$\alpha_{11}\xi_1 + \alpha_{12}\xi_2 = \lambda\xi_1 \tag{3.16a}$$

which, by using (3.14e) and (3.14g), is rewritten as

$$(A + HC_2)\xi_1 + (B_2 + HD_{22})(\gamma_1\xi_1 + \gamma_2\xi_2) = \lambda\xi_1. \tag{3.16b}$$

In view of (3.15a), the above equation reduces to

$$(A + HC_2)\xi_1 = \lambda\xi_1 \tag{3.16c}$$

which clearly establishes that the unobservable eigenvalue belongs to $\mathcal{P}_{\text{observer}}$.

Proceeding similarly, it can be shown that if (3.13) is uncontrollable, then the uncontrollable eigenvalue belongs to $\mathcal{P}_{\text{regulator}}$.

Note in the above development that $\xi_1 = 0$ contradicts the minimality assumption of $K(s)$. Thus,

$$\mathcal{P}_{\text{unobservable}} \subset \mathcal{P}_{\text{observer}} \tag{3.17a}$$

$$\mathcal{P}_{\text{uncontrollable}} \subset \mathcal{P}_{\text{regulator}} \tag{3.17b}$$

where $\mathcal{P}_{\text{unobservable}}$ is the set of all the unobservable poles of the controller $Q(s)$. $\mathcal{P}_{\text{uncontrollable}}$ is defined similarly. This completes the proof of Theorem 3-2.

IV. Conclusion

The poles of the closed-loop system with the observer-based controller parameterization shown in Fig. 4 can be classified into three groups, and each group of poles can be independently determined. These three groups of poles are the regulator poles (the eigenvalues of $A + B_2F$), the observer poles (the eigenvalues of $A + HC_2$), and the poles of the added dynamics $K(s)$. F, H, and $K(s)$ are free parameters to be chosen such that the closed-loop transfer function matrix $\Phi(s)$ has some optimal performance subject to the following constraints: $A + B_2F$ and $A + HC_2$ are stable and $K(s)$ is proper stable with $I - D_{22}K(\infty)$ invertible.

If the realization of the controller in Fig. 4 is not minimal, then the uncontrollable and/or unobservable controller poles can be removed and the order of the controller is minimized. The set of these removable controller poles is a subset of the regulator and the observer poles. The poles of the closed-loop system with the minimal order controller will include all the poles of the parameter matrix $K(s)$ and some of the regulator and the observer poles which are not the removable controller poles.

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Model Reduction of Linear Conservative Mechanical Systems

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***Abstract*—A new approach for model reduction of linear conservative or weakly damped mechanical systems is proposed which is based on the balancing of an associated gradient system.**

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I. Introduction

We consider linear conservative mechanical systems with external controls u, written in Hamiltonian form as

$$\begin{pmatrix} \dot q \\ \dot p \end{pmatrix} = \begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix} + \begin{pmatrix} 0 \\ B \end{pmatrix} u,$$

$$P = P^T > 0,\ Q = Q^T > 0 \quad \text{(1a)}$$

with $q = (q_1, \cdots, q_n)^T$ the vector of generalized configuration coordinates and $p = (p_1, \cdots, p_n)^T$ the vector of generalized momentums. The expressions $\frac{1}{2}p^T Pp$ and $\frac{1}{2}q^T Qq$ are the kinetic, respectively, potential, energy of the system. Although in applications the total energy $\frac{1}{2}p^T Pp + \frac{1}{2}q^T Qq$ is usually not strictly conserved (due to dissipation), often the conservative model (1a) serves as a useful starting point, especially if the inherent damping in the system is negligible and/or difficult to quantify. In principle, the output map for a system (1a) can be any function of q and p (or of q and $\dot q$ since $p = P^{-1}\dot q$); see [15]. However, in the present paper, we concentrate on the case of *collocated* sensors and actuators for which the outputs are given as

$$y = B^T q. \quad \text{(1b)}$$

System (1a), (1b) is called a *Hamiltonian system*, and is known to have some enjoyable properties (see, e.g., [3], [13], [14]). The transfer matrix $F_H(s)$ of (1) is given as [14]

$$F_H(s) = B^T(Is^2 + PQ)^{-1}PB, \quad (2)$$

and therefore satisfies the symmetry properties

$$F_H(-s) = F_H(s) = F_H^T(-s). \quad (3)$$

Conversely, it can be shown ([13]; see also [4]) that if a transfer matrix $F(s)$ satisfies (3), then there always exists a minimal realization of the form (1), with $\det P \neq 0$, but P and Q *not* necessarily positive definite. The assumption $P > 0$ and $Q > 0$ implies that the poles of the system are all on the imaginary axis and unequal to zero, and that the system is *marginally* (but not asymptotically) stable. In fact, it is well known (cf. [1]) that there always exists a state-space transformation

$$\bar q = Sq,\ \bar p = S^{-T}p,\ \det S \neq 0 \quad (4)$$

for (1) such that the system matrix $\begin{pmatrix} 0 & P \\ -Q & 0 \end{pmatrix}$ transforms into

$$\begin{pmatrix} 0 & I_n \\ -D & 0 \end{pmatrix}, \qquad D = \mathrm{diag}(\omega_1^2, \cdots, \omega_n^2), \quad (5)$$

from which it is clear that the motion of (1) for $u = 0$ decomposes into n independent eigenmodes with eigenfrequencies $\omega_1, \cdots, \omega_n$. This is called *modal analysis*.

The problem investigated in the present paper is that of *model reduction* of a Hamiltonian system (1). Most common approaches to model reduction are based on the above displayed modal analysis of the system matrix; indeed, the system is reduced to a simpler system by leaving out the eigenmodes corresponding to some of the eigenfrequencies (usually the higher ones). However, since modal analysis basically is only concerned with the system matrix, and *not* with the input and output matrix, the resulting model reduction may have disadvantages from a system and control theoretic point of view. For instance, it is clear that the omission of a particular high eigenfrequency for which the corresponding input component happens to be large will result in substantial control (and observation) spillover (cf. [2]). In this paper, we therefore wish to give an alternative approach using the *joint* knowledge of the system matrix and the input and output matrices of (1). This approach is heavily motivated by the well-established technique of model reduction of asymptotically stable systems using *balancing* (cf. [6], [10], [12]). Let us stress here that the balancing procedure itself cannot be applied to systems (1) since (1) is only *marginally* and *not* asymptotically stable, and hence the controllability and observability gramians (cf. [10], [12]) for (1) cannot be defined as improper integrals. Furthermore, also, an alternative definition of these gramians as unique solutions of some well-defined Riccati equations (cf. [9]) is easily seen not to be feasible in this case. This problem was partially circumvented in [7], [8] by balancing weakly damped (and therefore asymptotically stable) mechanical systems, and by carrying out a limiting analysis for infinitely small damping. Although appealing, it is not clear if this approach is the most natural one; moreover, the numerical problems related to the computations of the gramians of a weakly damped mechanical system seem not to be resolved [5], [7]. For other interesting approaches based on "closed-loop" balancing, respectively, modal cost analysis, we refer to [11], [17], respectively [16].

The key idea of the approach presented in this paper is to associate as in [3], [13] with the Hamiltonian system (1) the *gradient* (or reciprocal) system

$$\dot x = -PQx + PBu, \qquad P = P^T > 0,\ Q = Q^T > 0 \quad \text{(6a)}$$

$$y = B^T x \quad \text{(6b)}$$

or, equivalently,

$$P^{-1}\dot x = -Qx + Bu, \qquad P = P^T > 0,\ Q = Q^T > 0 \quad \text{(6a}'\text{)}$$

$$y = B^T x \quad \text{(6b}'\text{)}$$

with inner product P^{-1} and potential function $\frac{1}{2}x^T Qx$. (If the Hamiltonian system (1) is physically realized by masses and springs, then the gradient system (6) is obtained by replacing the springs by dampers.)

The above transition from Hamiltonian to gradient system is a *basis-free* operation, as already explained in [3], [14]. Indeed, a basis transformation $\bar q = Sq$, $\det S \neq 0$ induces the symplectic transformation (4), transforming the Hamiltonian system (1) into

$$\begin{pmatrix} \dot{\bar q} \\ \dot{\bar p} \end{pmatrix} = \begin{pmatrix} 0 & SPS^T \\ -S^{-T}QS^{-1} & 0 \end{pmatrix} \begin{pmatrix} \bar q \\ \bar p \end{pmatrix} + \begin{pmatrix} 0 \\ S^{-T}B \end{pmatrix} u \quad (7)$$

$$y = B^T S^{-1}\bar q.$$

The associated gradient system of (7) is given as

$$(S^{-T}P^{-1}S^{-1})\dot{\bar x} = -S^{-T}QS^{-1}\bar x + S^{-T}Bu$$

$$y = B^T S^{-1}\bar x \quad (8)$$

and thus is obtained from (6′) by the state-space transformation $\bar x = Sx$. (Notice that the inner product P^{-1} and potential function $\frac{1}{2}x^T Qx$ both transform in the right, covariant way.)

Clearly, the transfer matrix $F_G(s)$ of (6) is related to $F_H(s)$ [see (2)] as

$$F_H(s) = F_G(s^2) \quad (9)$$

and thus satisfies $F_G(s) = F_G^T(s)$. (Conversely, it can be shown, see, e.g., [18], that a transfer matrix $F(s)$ satisfying $F(s) = F^T(s)$ always has a minimal realization of the form (6), with $\det P \neq 0$ but P and Q not necessarily positive definite.) Also, the following relation is easily proven.

Proposition 1 [13], [14]: The Hamiltonian system (1) is controllable (respectively, observable) if and only if its associated gradient system (6) is controllable (respectively, observable). Furthermore (by collocation of sensors and actuators), the Hamiltonian (respectively, gradient) system is controllable iff the Hamiltonian (respectively, gradient) system is observable.

Indeed, as already remarked in [3], there are physical reasons which suggest that controllability (observability) of the Hamiltonian system (1) is closely related to controllability (observability) of the associated gradient system (6). Since model reduction by balancing is based on exploiting the controllability and observability properties of the system, this partly motivates the consideration of the associated gradient system for model reduction of the Hamiltonian system.

From now on, we shall assume that the Hamiltonian system (1) is *minimal*. Since, as we have seen, the poles of $F_H(s)$ are all purely imaginary and nonzero, it then immediately follows from (9) and Proposition 1 that the gradient system (6) is minimal and all its eigenvalues are all real and strictly negative. In particular, the associated gradient system is *asymptotically stable*.

II. Model Reduction of Hamiltonian Systems via Pseudobalancing

Since the associated gradient system (6) is asymptotically stable, we can apply balancing [10] to it. This is done by comparing the *controllability gramian* W of (6), defined as the unique solution of the Riccati equation

$$(-PQ)W + W(-PQ)^T = -(PB)(PB)^T \tag{10}$$

to its *observability gramian* M, defined as the unique solution of the Riccati equation

$$(-PQ)^T M + M(-PQ) = -BB^T. \tag{11}$$

If we apply a coordinate transform $\bar{x} = Sx$ to the gradient system (6), then the gramians W and M transform as

$$W \to SWS^T, \quad M \to S^{-T}MS^{-1}. \tag{12}$$

It follows from the theory of balancing [10], [12] that there always exist coordinate transformations $\bar{x} = Sx$ such that

$$SWS^T = S^{-T}MS^{-1} = \operatorname{diag}(\sigma_1, \cdots, \sigma_n) =: \Sigma \tag{13}$$

with $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$. [Actually, the argument is similar to the one underlying the existence of a transformation (4) resulting in (5).] Furthermore, $\sigma_1, \cdots, \sigma_n$ are input-output invariants, and are given as the square roots of the eigenvalues of MW. Let us call them the *singular values* of the system (6). The system in such coordinates satisfying (13) is said to be *balanced*. Model reduction of a balanced system is achieved by omitting the last $n-k$ state-space variables corresponding to the singular values $\sigma_{k+1}, \cdots, \sigma_n$ where $\sigma_k \gg \sigma_{k+1}$ (see [10], [12]). In the present case of balancing of gradient systems (6), we additionally have the following.

Lemma 2: The controllability and observability gramians W and M for (6) are related as

$$W = PMP. \tag{14}$$

Proof: W and M are given as the unique solutions of (10), respectively, (11). On the other hand, pre- and postmultiplication of (11) with P yields $-PQPMP - PMPQP = -PBB^TP$, and hence, PMP also satisfies (10). □

Proposition 3: Let (6) be balanced. Then

$$P = I_n. \tag{15}$$

Furthermore, suppose $\sigma_k > \sigma_{k+1}$, and write

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \tag{16}$$

with $\dim x_1 = k$, and Q_{11} a $k \times k$ and B_1 a $k \times m$ matrix. Then the reduced system

$$\dot{x}_1 = -Q_{11}x_1 + B_1 u$$
$$y = B_1^T x_1 \tag{17}$$

is a gradient system which is minimal, balanced, and asymptotically stable with $Q_{11} > 0$.

Proof: Since the system is balanced, we have $M = W = \Sigma$ with Σ diagonal. By Lemma 2, this implies

$$\Sigma = P\Sigma P. \tag{18}$$

Since the ith row of ΣP equals the ith row of the matrix P premultiplied with the positive factor σ_i, it follows that the product of the jth column of P with the ith row of P equals the zero matrix for $i \neq j$. Hence, PP is diagonal, and in fact, $P^2 = I_n$. Since $P > 0$, the eigenvalues of P are real and positive. Together with $P^2 = I_n$, this implies that the eigenvalues of P are all 1, and thus $P = I_n$. Clearly, (17) is a gradient system. The fact that (17) is minimal, balanced, and asymptotically stable follows from standard balancing theory [12]. Finally, since $Q > 0$, $Q_{11} > 0$ also. □

Remark: A gradient system (6) for which $P = I_n$ and $Q > 0$ is called a *relaxation system* in [18]. It thus follows from Proposition 3 that a balanced gradient system is always a relaxation system. This can be seen as the analog of [12, Theorem 3.1]. (A relaxation system has purely nonoscillatory behavior converging to the origin.) A related result for single-input, single-output systems for P indefinite was obtained in [7].

Now let us see how this translates to the original Hamiltonian system (1). As we saw before [cf. (7), (8)], a basis transformation $\bar{x} = Sx$ for the gradient system (6) corresponds to the *symplectic* transformation (4) for the Hamiltonian system (1), transforming it into (7).

In particular, if $\bar{x} = Sx$ is a coordinate transformation which brings the gradient system (6) into balanced form, then by Proposition 3 [cf. (15)], $SPS^T = I_n$, and thus the induced symplectic transformation (4) transforms the Hamiltonian system (1) into

$$\begin{pmatrix} \dot{\bar{q}} \\ \dot{\bar{p}} \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -\bar{Q} & 0 \end{pmatrix} \begin{pmatrix} \bar{q} \\ \bar{p} \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{B} \end{pmatrix} u$$
$$y = \bar{B}^T \bar{q} \tag{19}$$

where $\bar{Q} := S^{-T}QS^{-1}$, $\bar{B} := S^{-T}B$. We shall call (19) a *pseudobalanced* Hamiltonian system. Formally, we have the following.

Definition 4: The Hamiltonian system (1) is said to be *pseudobalanced* if the associated gradient system is balanced.

Model reduction of a pseudobalanced Hamiltonian system (19) will now be based on model reduction of the associated balanced gradient system. Indeed, suppose that the singular values $\sigma_1 \geq \cdots \geq \sigma_n$ of the gradient system satisfy $\sigma_k \gg \sigma_{k+1}$, and write [compare (16)]

$$\bar{q} = \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}, \quad \bar{p} = \begin{pmatrix} \bar{p}_1 \\ \bar{p}_2 \end{pmatrix}, \quad \bar{Q} = \begin{pmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \end{pmatrix} \tag{20}$$

with $\dim \bar{q}_1 = \dim \bar{p}_1 = k$, $\bar{Q}_{11}$ a $k \times k$ matrix, and $\bar{B}_1$ a $k \times m$ matrix. Then a reduced-order model of (19) is given as

$$\begin{pmatrix} \dot{\bar{q}}_1 \\ \dot{\bar{p}}_1 \end{pmatrix} = \begin{pmatrix} 0 & I_k \\ -\bar{Q}_{11} & 0 \end{pmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{p}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ \bar{B}_1 \end{pmatrix} u$$
$$y = \bar{B}_1^T \bar{q}_1. \tag{21}$$

Proposition 5: Suppose $\sigma_k > \sigma_{k+1}$ for the pseudobalanced Hamiltonian system (19). Then (21) is a Hamiltonian system which is pseudobalanced, minimal, and marginally stable with $\bar{Q}_{11} > 0$.

Proof: Clearly, (21) is Hamiltonian (with dimension of its state space equal to $2k$). By Proposition 3, the associated gradient system of (21) is balanced, minimal, and has $\bar{Q}_{11} > 0$. Thus, (21) is pseudobalanced (Definition 4), minimal (Proposition 1), and the total energy $\frac{1}{2}\bar{p}_1^T\bar{p}_1 + \frac{1}{2}\bar{q}_1^T\bar{Q}_{11}\bar{q}_1$ is positive, implying that (21) is marginally stable. □

Recall that the eigenfrequencies $\omega_1, \cdots, \omega_n$ of the full-order system (19) are obtained by calculating an orthonormal matrix R such that $R\bar{Q}R^T = \operatorname{diag}(\omega_1^2, \cdots, \omega_n^2)$ [indeed, the symplectic transformation (4) with S replaced by R will transform the system matrix of (19) into the form (5)], and thus equal the square roots of the eigenvalues of $\bar{Q}$. Similarly, the eigenfrequencies $\omega_1^r, \cdots, \omega_k^r$ of the reduced-order system (21) are the square roots of the eigenvalues of $\bar{Q}_{11}$. We immediately obtain the following relationship.

Proposition 6: The eigenfrequencies of the system (19) and (21) satisfy

$$\min_{i \in n} \omega_i \leq \min_{i \in k} \omega_i^r, \quad \max_{i \in k} \omega_i^r \leq \max_{i \in n} \omega_i. \tag{22}$$

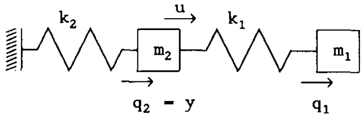

Fig. 1.

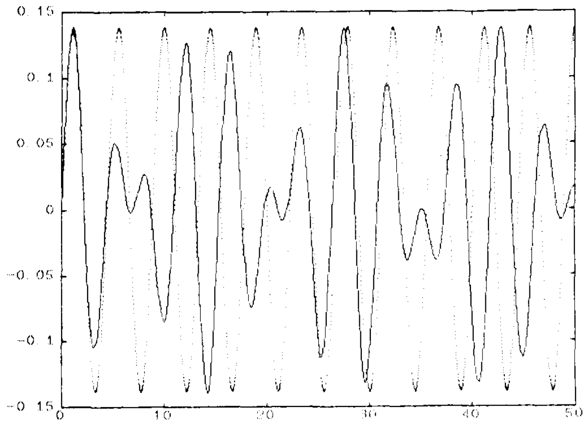

Fig. 2. Impulse responses of the fourth-order system and of (24) (dotted line).

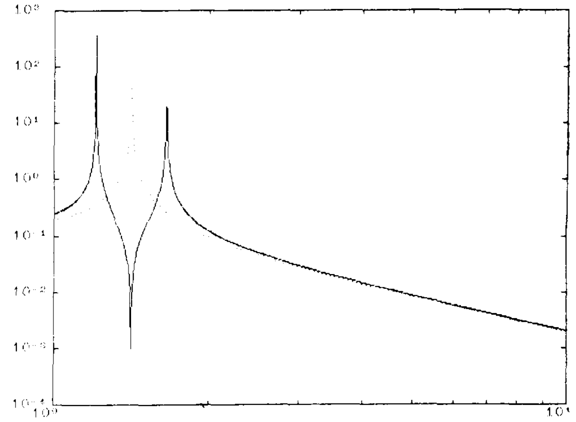

Fig. 3. Bode amplitude plots of the fourth-order system and of (24) (dotted line).

Proof: The first inequality follows from

$$\min_{i\in n}\omega_i^2 = \min_{|\bar{x}|=1} \bar{x}^T\bar{Q}\bar{x},\ \min_{i\in k}(\omega_i^r)^2 = \min_{|x_1|=1} \bar{x}_1^T\bar{Q}_{11}\bar{x}_1$$

$$= \min_{|\bar{x}|=1, x_2=0} \bar{x}^T\bar{Q}\bar{x} \tag{23}$$

and similarly for the second inequality. □

Note that, contrary to the model reduction by modal analysis, the set of eigenfrequencies $\omega_1^r, \cdots, \omega_k^r$ of the reduced system generally will *not* be a subset of the set of eigenfrequencies of the original system (in fact, for $m < n$, this will never be the case).

III. An Example

The fundamental question raised by the previous section is in what sense the reduced-order Hamiltonian system (21) is a good approximation to the full-order Hamiltonian system (19) or (1). Up to now, we do not have a full theoretical answer to this question. In all numerical examples we tried so far, it was found that the reduced-order models obtained with our approach give approximations to the full-order system which are usually "better" and never "worse" than the approximations resulting from the reduced-order models obtained by modal analysis (by leaving out some of the eigenmodes of the system). Here, "better" has to be understood in the naive sense of impulse responses and Bode plots being closer to the corresponding figures for the full-order system.

In order to show the main characteristics of our method, we now give a simple toy example of a mechanical system consisting of two masses m_1, m_2 attached to springs with spring constants k_1, k_2 (see Fig. 1).

The control u acts on mass m_2, and the position of mass m_2 (relative to rest position) is observed as output y. The impulse response and the Bode plot (amplitude) of the system are depicted in Figs. 2 and 3.

The singular values of the associated gradient system are computed as $\sigma_1 = 0.0488$, $\sigma_2 = 0.0012$. It is clear from the impulse response of the system (Fig. 2) that no second-order system will satisfactorily approximate the full fourth-order system. This conclusion is enforced by the fact that the singular values of the Hamiltonian system should be taken as $\sqrt{\sigma_1}$ and $\sqrt{\sigma_2}$, and since $\sqrt{\sigma_1} \approx 6.3\sqrt{\sigma_2}$, these singular values are too close to each other to have a good model reduction. However, if we *do* apply pseudobalancing and leave out the components corresponding to σ_2, then we obtain the reduced-order system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -1.9952 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0.4415 \end{pmatrix} u,\ y = (0.4415 \quad 0)x \tag{24}$$

which has an impulse response and Bode diagram as depicted in Figs. 2 and 3 (dotted lines), and which reasonably approximates the fourth-order system.

On the other hand, if we would reduce the model by modal analysis, then we obtain a second-order system with frequency equal to the first or the second eigenfrequency of the system, and with amplitude in both cases approximately 0.7, which clearly is very unsatisfactory. Notice also that modal analysis does not *a priori* tell us which eigenmode has to be left out, contrary to our pseudobalancing approach which is based upon the calculation of the singular values.

IV. Conclusion

We have given an alternative approach to model reduction of a fairly large class of linear conservative mechanical systems, which in numerical examples looks promising. If we would have a general dependence of the outputs of the configuration coordinates, i.e., instead of (1b) an equation $y = Cq$, then our procedure could still be applied, *mutatis mutandis*, but the basic Proposition 5 will not hold anymore in this noncollocated case.

The approach taken in this paper raises two fundamental questions. First, we do not yet know what kind of "norm" is underlying our approach of *pseudobalancing* and reducing a Hamiltonian system (contrary to the balancing procedure for asymptotically stable systems, which is related to the Hankel norm of the system). Possibly, a clue of this problem is the following observation, due to an anonymous referee. The mapping $s \to s^2$ maps the 45° radials in the complex plane onto the imaginary axis. In view of (9), this implies that transfer matrix error bounds along the imaginary axis for the reduced gradient system (see, e.g., [6]) translate into error bounds along the 45° radials for the reduced Hamiltonian system. Second, it would be nice to have some *physical* interpretation of the reduced-order model (21). Finally, our approach can be also applied to nearly conservative (weakly damped) systems by first leaving out the damping in the model reduction process, and then adding it again to the reduced-order model, and to infinite-dimensional (nearly) conservative systems by first reducing the system to a high- but finite-dimensional (nearly) conservative system by modal analysis (omitting the high eigenfrequencies) (compare [5]).

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Generalized Functional Observer

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Abstract—**In this paper, we propose a functional observer and state feedback for singular systems in the polynomial fraction form that requires no prerequisite impulsive mode elimination. The order of the compensator is determined by the newly defined generalized observability index that is associated with the McMillan degree of the system. A new generalized Lyapunov equation is also proposed through a realization scheme that can be applied to both ordinary and singular systems. The solution to the equation provides an algebraic approach to the observer of singular systems in the generalized state-space form.**

I. Introduction

Singular systems may be expressed either in the generalized state-space form

$$\bar{G}(s) = C(sE - A)^{-1}B \tag{1}$$

where E is singular, or in the polynomial fraction form, for instance, in a right polynomial fraction form

$$\bar{G}(s) = P(s)Q^{-1}(s) \tag{2}$$

where at least one of $Q^{-1}(s)$ and $P(s)Q^{-1}(s)$ is improper. If E in (1) is nonsingular or both $Q^{-1}(s)$ and $P(s)Q^{-1}(s)$ in (2) are proper, then the system is called ordinary. The observer design for singular systems is different from the one for ordinary systems. For the generalized state-space form (1), this problem has been discussed in [9], and the dual situation of the eigenvalue assignment has been discussed in [6] by forming a generalized Lyapunov equation.

In this paper, we shall present the functional observer design for singular systems in the right polynomial fraction form (2). We shall show that if the system (2) is strongly observable, i.e., there exist neither finite nor infinite output decoupling zeros, then there exists a causal functional observer; the reconstructed state can then be used for the arbitrary eigenvalue assignment, noting that the system (2) is always strongly controllable [1]. The design requires no prerequisite step to eliminate the impulsive modes. We shall also show that the order of the compensator is determined by the newly defined generalized observability index. It is interesting to note that while the ordinary observability index is associated with the order of the system, the generalized observability index of a singular system is associated with the McMillan degree of the system.

In the functional observer design, we use a generalized realization scheme that can be applied to both ordinary and singular systems, which is called the canonical form in this paper. By using the scheme for both the singular plant and the ordinary observer, the observer problem is formulated in a new generalized Lyapunov equation for singular systems in the generalized state-space form (1). The equation will be solved through a Diaphantine-type equation in terms of the functional observer for the polynomial fraction form (2). Without loss of generality, this provides an algebraic approach to the observer of singular systems. From the computation point of view, if the system is given in the polynomial fraction form or is numerically transferrable to the canonical form, then the design proposed in this paper can directly apply. Since the reduction of a generalized state-space form to the canonical form may be numerically unstable, the research for the observer design in the generalized state-space form is still of great interest.

In the following sections, we first present the generalized realization scheme and the definition of the generalized observability index. Then we show the functional observer and state feedback design for singular systems in the polynomial fraction form (2). In the last section, we give the formulation of the generalized Lyapunov equation and show the solutions.

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II. Generalized Realization Scheme and the Generalized Observability Index

Consider the system (2). Note that the system is always strongly controllable, e.g., see [1]. The strong observability should, however, be determined by tests. The following is a convenient one.

Lemma 1 [14]: Consider the polynomial matrix $[Q'(s)P'(s)]'$. If there exists no infinite output decoupling zero, then for any unimodular matrix $U(s)$ such that $[Q'(s)P'(s)]'U(s)$ is column reduced, we have

$$\delta_{ci}U(s) \le \delta_{ci}[Q'(s) \quad P'(s)]'U(s), \qquad \text{for all } i \tag{3}$$

where δ_{ci} denotes the ith column degree of the polynomial matrix. □

The definition of the infinite decoupling zeros of the polynomial matrix follows the one in [8] rather than the ones in [7]. The test in Lemma 1 implies that if the condition is satisfied, then there exist no infinite output decoupling zeros in both $P(s)Q^{-1}(s)$ and $P(s)U(s)(Q(s)U(s))^{-1}$. Therefore, under this condition, a class of unimodular matrix operations can be applied to let the matrix $[Q'(s) \; P'(s)]'$ be column reduced without affecting the system property of infinite output decoupling zeros. Without loss of generality, we assume in the following content that the matrix $[Q'(s) \; P'(s)]'$ is column reduced with the ith column degree μ_i. Also, we assume that $Q(s)$ and $P(s)$ are right coprime. Therefore, the system (2) is strongly observable.

Let us write

$$\begin{bmatrix} Q(s) \\ P(s) \end{bmatrix} = \begin{bmatrix} Q_0 \\ P_0 \end{bmatrix} \operatorname{diag}\{s^{\mu_i}\} + \begin{bmatrix} Q_1 \\ P_1 \end{bmatrix} \operatorname{diag}\{s^{\mu_i - 1}\} + \cdots + \begin{bmatrix} Q_\mu \\ P_\mu \end{bmatrix} \operatorname{diag}\{s^{\mu_i - \mu}\} \tag{4}$$

where

$$\mu = \max\{\mu_i\}. \tag{5}$$